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A Continuity Condition for the Existence of a Continuous Selection for a Set-Valued Mapping

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1. INTRODUCTION

In this paper I consider a set-valued mapping and give a sufficient condition in order that almost lower semicontinuity of this mapping be equivalent to the existence of a continuous selection. In Section 3, I prove that the metric projection onto a finite-dimensional subspace in $C(T)$, T compact, has this property and, thus, a question of Deutsch and Kenderov [5, 6] is answered.

This is probably the first step toward an intrinsic characterization of those finite-dimensional subspaces of $C(T)$ for which the metric projection admits a continuous selection. Special results were earlier obtained by Lazar, Morris and Wulbert [7], who characterized one-dimensional subspaces, and Nürnberger and Sommer (see [10]), who gave a characterization in the case $T = [a, b]$.

It turns out that most of the essential ideas I use were implicit in a paper of Blatter and Schumaker [2], who proved some uniqueness results for continuous selections. However, it can be easily seen that a weaker condition than continuity of the selection was actually used in their proofs.

2. THE SET OF CONTINUOUS SELECTIONS

Let X be a topological space, Y a metric space with metric d, 2^Y the collection of all non-empty subsets of Y. Let $F: X \to 2^Y$ a mapping whose images are non-empty subsets of Y. A selection S for F is a mapping $S: X \to Y$ such that $S(x) \in F(x)$ for each $x \in X$.

We define the set of continuous selections for F by setting

$$\tilde{F}(x) := \{S(x) \mid S \text{ is a continuous selection for } F\}, \qquad x \in X.$$

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It is obvious that

PROPOSITION 2.1. *There exists a continuous selection for F iff $\tilde{F}(x) \neq \emptyset$ for each $x \in X$.*

The question of whether or not F admits a continuous selection is now equivalent to whether or not these sets $\tilde{F}(x)$, $x \in X$, are non-empty.

To obtain a result I recall a definition of Brown [1]. For $y \in Y$ and $A \in 2^Y$, the distance from y to A is denoted by

$$d(y, A) := \inf\{d(y, a) \mid a \in A\}$$

and then

$$F'(x) := \{y \in F(x) \mid d(y, F(z)) \to 0 \text{ as } z \to x\}.$$

PROPOSITION 2.2. *Assume $F'(x) \subset \tilde{F}(x)$ for each $x \in X$. Then F admits a continuous selection iff $F'(x) \neq \emptyset$ for each $x \in X$.*

Proof. Obviously we have for each $x \in X$, $\tilde{F}(x) \subset F'(x)$, hence $\tilde{F}(x) = F'(x)$ for each $x \in X$. By Proposition 2.1, F admits a continuous selection iff $F'(x) \neq \emptyset$ for each $x \in X$.

The ε-neighborhood of a non-empty set $A \subset Y$ is given by

$$B_\varepsilon(A) := \{y \in Y \mid d(y, A) < \varepsilon\}.$$

Deutsch and Kenderov gave the following definition [5].

DEFINITION 2.3. F is called *almost lower semicontinuous* (alsc) (resp. *n-lower semicontinuous* (n-lsc)) at $x_0 \in X$ iff for each $\varepsilon > 0$ there exists a neighborhood U of x_0 such that

$$\bigcap_{x \in U} B_\varepsilon(F(x)) \neq \emptyset$$

(resp. $\bigcap_{i=1}^n B_\varepsilon(F(x_i)) \neq \emptyset$ for each choice of n points $x_1, ..., x_n$ in U). F is called *almost lower semicontinuous* (alsc) (resp. *n-lower semicontinuous* (n-lsc)) iff F is alsc (resp. n-lsc) at each point of X.

For a compact-valued mapping this definition was characterized by Deutsch, Indumathi and Schnatz [4].

LEMMA 2.4. *Let $x \in X$ and assume that $F(x)$ is compact. Then F is alsc at x iff $F'(x) \neq \emptyset$.*

We get easily

PROPOSITION 2.5. *Assume $F(x)$ is compact and $F'(x) \subset \tilde{F}(x)$ for each $x \in X$. Then F admits a continuous selection iff F is alsc.*

The question is now, which compact-valued mappings have the property that $F'(x) \subset \tilde{F}(x)$ for each $x \in X$? Unfortunately not every compact-valued mapping has this property, as a counterexample of Pelant shows (see [5]).
The following three theorems concerning almost lower semicontinuity were proved by Deutsch and Kenderov [5].

THEOREM 2.6. *Let X be a paracompact space and let Y be a normed linear space. Let $F: X \to 2^Y$ have convex images. Then F is alsc iff for each $\varepsilon > 0$, F has a continuous ε-approximate selection (i.e., a continuous mapping $S: X \to Y$ such that $S(x) \in B_\varepsilon(F(x))$ for each $x \in X$).*

Using a theorem of Helly, they proved:

THEOREM 2.7. *Let Y be an n-dimensional normed linear space, and suppose the mapping $F: X \to 2^Y$ has closed, bounded, and convex images. Then F is $(n+1)$-lsc iff F is alsc.*

And they characterized, for one-dimensional spaces, when a continuous selection exists.

THEOREM 2.8. *Let X be a paracompact space, Y a one-dimensional normed linear space and suppose the mapping $F: X \to 2^Y$ has closed, bounded, and convex images. Then F has a continuous selection iff F is 2–lsc.*

3. THE METRIC PROJECTION IN $C(T)$

In this section we consider the space $C(T)$ of all continuous, real-valued functions on a compact Hausdorff space T, endowed with the usual uniform norm. For an n-dimensional subspace G in $C(T)$ we call

$$P(f) := \{g \in G \mid \|f - g\| = d(f, G)\}, \qquad f \in C(T),$$

the set of all best approximations from f to G, where $d(f, G)$ is called the distance from f to G. It is well known that the metric projection $P: C(T) \to 2^G$ maps $f \in C(T)$ into a non-empty, convex, and compact subset of G.
The next definition is that of the lower semicontinuous kernel given by Blatter and Schumaker [2]. I call it here, briefly, the kernel, but the definition is exactly the same. Note that it is independent of continuity of the given selection.

DEFINITION 3.1. Assume S^* is a selection for P. We define the kernel P^* of P induced by S^* as follows: Fix $f \in C(T)$. Set $H_0 := P(f)$ and for $k = 1, 2, \ldots,$

$$H_k := \{g \in H_{k-1} \mid g \text{ coincides with } S^*(f) \text{ in some neighborhood of } E(f - H_{k-1})\},$$

where $E(f - H_{k-1}) := \bigcap_{g \in H_{k-1}} \{t \in T \mid |f(t) - g(t)| = d(f, G)\}$. It is easily verified that

for each $k = 1, 2, \ldots,$ there exists a neighborhood of $E(f - H_{k-1})$ in which all elements of H_k coincide with $S^*(f)$,
the sets H_k are all closed, convex, and $P(f) = H_0 \supset H_1 \supset H_2 \supset \cdots \supset \{S^*(f)\}$, and
for each $k = 1, 2, \ldots,$ if H_k is a proper subset of H_{k-1}, then $\dim(H_k) < \dim(H_{k-1})$.

It then follows that the sequence $H_0, H_1, H_2, \ldots,$ is stationary from some point on. Let $k \geq 1$ be the smallest integer for which $H_k = H_{k-1}$ and set $P^*(f) := H_{k-1}$.

DEFINITION 3.2. A selection S^* for P has property $(*)$ iff for each $f \in C(T)$, $S^*(f) \in P'(f)$ (i.e., $d(S^*(f), P(h)) \to 0$ as $h \to f$).

Note that P has a selection with property $(*)$ iff $P'(f) \neq \emptyset$ for all $f \in C(T)$, hence by Lemma 2.4 iff P is alsc. Note moreover, that every continuous selection has property $(*)$ but the converse is not true, though I will prove later that property $(*)$ implies the existence of a continuous selection.

In the subspace of polynomial m-splines with k simple knots in $C[a, b]$, Blatter and Schumaker [3] proved that there exists a unique maximal alternator iff $k \leq m$ and using a characterization result of Nürnberger and Sommer [9] they showed, among other results, that the selection consisting of this unique maximal alternator has property $(*)$ and is in general not continuous.

LEMMA 3.3. *Let S^* be a selection for P with property $(*)$ and let P^* be the kernel of P induced by S^*. Then for every $f \in C(T)$ and every $\varepsilon > 0$ there exists an $f_\varepsilon \in C(T)$ such that $\|f_\varepsilon - f\| < \varepsilon$ and $P(f_\varepsilon) \subset P^*(f)$.*

Proof. Blatter and Schumaker proved this in [2, Lemma 3] under the stronger asumption of a continuous selection. However, they only used property $(*)$ in their proof.

THEOREM 3.4. *Let S^* be a selection for P with property $(*)$ and let P^**

be the kernel of P induced by S^*. Then the kernel mapping $P^*: C(T) \to 2^G$ is lower semicontinuous.

Proof. The proof is a slight modification of the corresponding proof in the main theorem of [2]. We just have to replace $(S^*(\tilde{f}_n))$ by a sequence (g_n) converging to $S^*(f)$ such that $g_n \in P(\tilde{f}_n)$, $n \in \mathbb{N}$, which exists since S^* has property $(*)$.

THEOREM 3.5. *Let S^* be a selection for P with property $(*)$ and let P^* be the kernel of P induced by S^*. Then $\tilde{P}(f) = P'(f) = P^*(f)$ for each $f \in C(T)$.*

Proof. The inclusion $\tilde{P}(f) \subset P'(f)$ is obvious, $P'(f) \subset P^*(f)$ is an easy consequence of Lemma 3.3 and the definition of $P'(f)$. By Theorem 3.4 the kernel mapping is lower semicontinuous, hence with the selection theorem of Michael [8], P^* resp. P has a continuous selection. In this case it is well known that $\tilde{P}$ is the largest lower semicontinuous submap of P (see [4]), this implies $P^*(f) \subset \tilde{P}(f)$, $f \in C(T)$.

It would be of interest to give a direct proof of $P'(f) \subset \tilde{P}(f)$, $f \in C(T)$, without using the definition of the kernel.

Now, the last theorem proves a conjecture of Deutsch and Kenderov [6] in the special case of $C(T)$, T compact.

COROLLARY 3.6. *The metric projection P in $C(T)$ onto an n-dimensional subspace admits a continuous selection iff P is $(n+1)$-lsc.*

Proof. If P is $(n+1)$-lsc, then by Theorem 2.7, P is alsc. Hence P has a selection with property $(*)$, by Lemma 2.4. We apply Theorem 3.5 and Proposition 2.5 to obtain a continuous selection for P. The converse is trivial.

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Note added in proof. After completion of the manuscript, I received a paper of Li Wu [11], who independently obtained Corollary 3.6, using similar methods.

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